The geography of amenable subrelations of acyclic graphs and beyond

(R. Tucker-Drob, A. Chen and G. Terlov)

Lecture 1

In these lectures we are concerned with (non) amenability of measure-class-preserving (mcp) ctbl Borel equivalence relations (CBERs) which will typically be treeable.

Measure-class-preserving (mcp) CBERs.

An eq. rel. E on a standard Borel space X (say $X := \mathbb{R}$) is called a CBER if each E-class is ctbl and E is a Borel subset of X^2.

$$\begin{tikzcd} \boxed{\text{CBERs}} \arrow[r, shift right, "\text{Feldman-Moore}"'] & \boxed{\text{Borel actions of ctbl groups}} \arrow[l, shift right] \end{tikzcd}$$

When E is measured, i.e. X is equipped with a Borel prob. measure μ, we say that E is measure-preserving (pmp) (resp. measure-class-preserving (mcp)) if $\forall \varphi \in [E] :=$ all Borel bijections $X \to X$ with graphs $\subseteq E$, $\varphi_* \mu = \mu$ (resp. $\varphi_* \mu \sim \mu$).
In fact, mcp $\iff$ the E-saturation $[Z]_E := \bigcup_{z \in Z} [z]_E$ of each null set $Z \subseteq X$ is null.

Every mcp CBER E admits a unique (up to null sets) Borel function
$w : E \to \mathbb{R}^+$, called the Radon-Nikodym cocycle (of E wrt μ),
$(x,y) \mapsto w^y(x) := \dfrac{\text{weight}(x)}{\text{weight}(y)}$, that

(i) is a cocycle: $w^x(y) \cdot w^y(z) = w^x(z)$, i.e. $\dfrac{\cancel{\text{weight}(y)}}{\text{weight}(x)} \cdot \dfrac{\text{weight}(z)}{\cancel{\text{weight}(y)}} = \dfrac{\text{weight}(z)}{\text{weight}(x)}$

(ii) and it satisfies the Mass Transport Principle: $\forall$ Borel $f : E \to [0, \infty]$,

$$\underbrace{\int \sum_{y \in [x]_E} f(x,y) \, d\mu(x)}_{\substack{\text{the outgoing mass} \\ \text{from } x \\ \text{(relative to } x)}} = \underbrace{\int \sum_{y \in [x]_E} f(y,x) \, w^x(y) \, d\mu(x)}_{\substack{\text{the incoming mass} \\ \text{to } x \\ \text{(relative to } x)}}$$

Exercise 1. (a) Show that (ii) $\Leftrightarrow$ (ii') $\forall \varphi \in [E]$, $\frac{d\varphi_* \mu}{d\mu}(x) = w^x(\varphi^{-1}(x))$.

(b) E is pmp $(\forall \varphi \in [E], \varphi_* \mu = \mu) \Leftrightarrow w \equiv 1$.

Treeable CBERs.

A graphing of a CBER E is a Borel graph G on X (i.e. $G \subseteq X^2$ Borel symmetric) such that the G-connected components are exactly the E-classes, i.e. the connectedness eq. rel. $\mathbb{E}_G$ of G is exactly E.

Example. Let E be the orbit eq. rel. of a Borel action of a ctbl gp Γ on X and let $\Gamma = \langle S \rangle$, where $S \subseteq \Gamma$ is symmetric.
The Schreier graph G_S of this action (wrt S) is the graph on X defined by:

$$(x,y) \in G_S \Leftrightarrow s \cdot x = y \text{ for some generator } s \in S.$$

Clearly, G_S is a graphing of E and if the action is free, then each component is a copy of the Cayley graph of Γ.

A CBER E is treeable if it admits a acyclic graphing, called a treeing. If μ is a Borel prob. measure on X, then E is called μ-treeable if it is treeable after throwing out an E-invariant Borel null set.

Example. Every free Borel action of the free group $\mathbb{F}_n$ (on $n \leq \omega$ generators) induces a treeable CBER; in fact, the standard Schreier graph is a treeing.

Treeable CBERs play the same role among all CBERs as free groups among all ctbl gps.

Hyperfinite/amenable CBERs.

A CBER E is hyperfinite if $E = \bigcup\limits_n^{\uparrow} E_n$, where E_n is a finite Borel eq. rel. (each class is finite). E is μ-hyperfinite if it is hyperfinite on an E-invariant Borel conull set.

Fact (Weiss, Slaman–Steel). A CBER is hyperfinite $\Leftrightarrow$ it's induced by a Borel action of $\mathbb{Z}$.

In particular, hyperfinite $\Rightarrow$ treeable.

A CBER E on (X, μ) is μ-amenable if there is a "μ-measurable" assignment $x \mapsto m_x$ of a finitely additive prob. measure on $[x]_E$, $x \in X$, that is E-invariant. "μ-measurable" means that for any Borel $B \subseteq X$, the map $\begin{array}{l} X \to \mathbb{R} \\ x \mapsto m_x(B) \end{array}$ is μ-measurable.

Fact (see Jackson–Kechris–Louveau)

(a) Mokobodzki: CBERs induced by Borel actions of amenable gps are μ-amenable.

(b) Folklore: Conversely, if a free pmp action $\Gamma \curvearrowright (X, \mu)$ induces a μ-amenable CBER, then Γ is amenable. (This is false outside of pmp.)

Connes-Feldman-Weiss. μ-hyperfinite = μ-amenable.

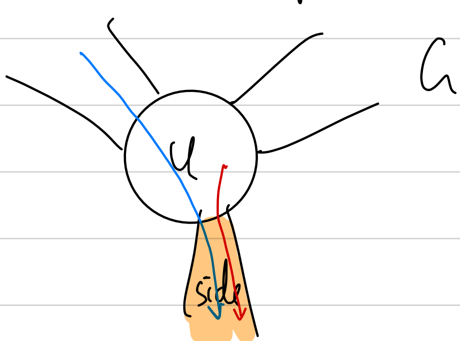
We will be working in the measure context, ignoring null sets, so for us hyperfinite = amenable.

Ends of graphs.

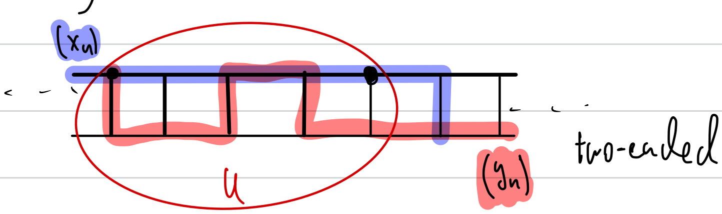
Let G be a connected graph on a vertex set V.

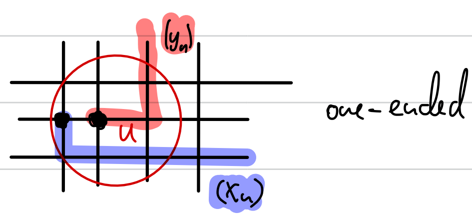
- A G-ray is an infinite simple G-path (v_n), and we denote the set of G-rays by $\mathrm{Rays}(G)$.

- For a set $U \subseteq V$ of vertices, a side of U is a connected component of the graph $G - U$ obtained from G by removing U.



- Let $\sim_G$ denote the end equiv. rel. on $\mathrm{Rays}(G)$, i.e. $(x_n) \sim_G (y_n)$ if for each finite set $U \subseteq V$ of vertices, (x_n) and (y_n) are eventually on the side of U.





- An end of G is just a $\sim_G$-eq. class. We denote the space of ends by $\partial_G V := \mathrm{Rays}(G) / \sim_G$.

- For a side S of a finite set $U \subseteq V$, let $\partial_G S :=$ all ends whose representative rays are eventually in S. Call $\bar{S}^G := S \cup \partial_G S$ an extended side of U.

- Let $\bar{V}^G := V \cup \partial_G V$ be equipped with topology generated by the vertex singletons and extended sides of finite vertex cuts. This is 0-dim and when G is locally cfbl, it's Polish.

When G isn't connected, all the above notions make sense, but the top. on $\bar{V}^G$ is no longer nice.

Adams' dichotomy for pmp.

We call a Borel graph G amenable if such is its connectedness relation $\mathbb{E}_G$. Here we characterize all treeings that are amenable in the pmp context.

Theorem (Adams). A pmp treeing is amenable $\iff$ it has ≤ 2 ends in a.e. connected component.

This fails outside of pmp, e.g. the boundary action $\mathbb{F}_2 \curvearrowright \partial \mathbb{F}_2$ is free on a coctbl set, so it admits a 4-regular treeing, yet the orbit eq. rel. is amenable (in fact, Borel hyperfinite).

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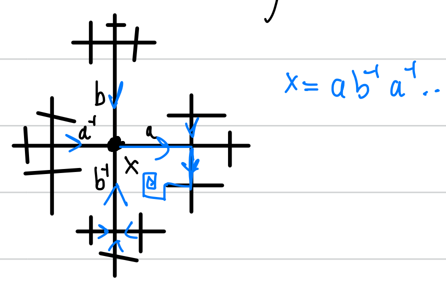
Lecture 2

Counterexample to Adams' in mcp. Let $\mathbb{F}_2 = \langle a, b \rangle$ be the free group on $\{a,b\}$. By the boundary of $\mathbb{F}_2$ we mean the set $\partial \mathbb{F}_2$ of all infinite reduced words in $\{a^{\pm 1}, b^{\pm 1}\}$, so it's a closed subset of $\{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{N}}$, i.e. words with no cancellation. $\mathbb{F}_2$ naturally acts on this by concatenation and cancellation, namely: if $s \in \{a^{\pm 1}, b^{\pm 1}\}$ and $w \in \partial \mathbb{F}_2$, then $s \cdot w := sw$ if $w \neq s^{-1} w'$ and otherwise, $s \cdot w = \cancel{s} \cdot (\cancel{s^{-1}} w') = w'$. This is a continuous action and it is free except on a ctbl set. Thus, this action admits a 4-regular treeing (with trees copies of the Cayley graph of $\mathbb{F}_2$):

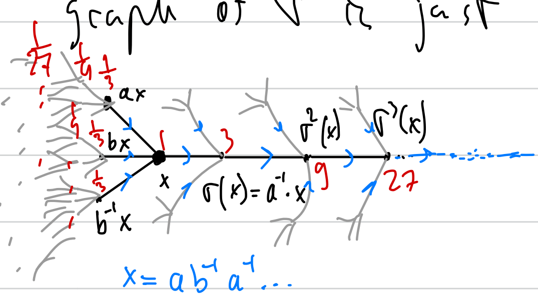


In particular, there are ∞-many ends.

Nevertheless, this action is hyperfinite.

We show this by showing that the orbit eq. rel. $E_{\mathbb{F}_2}$ is also induced by a single Borel ctbl-to-one function (and applying the result of Dougherty–Jackson–Kechris which says that such CBERs are hyperfinite).

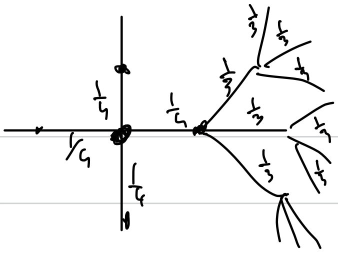
Let $\sigma : \partial \mathbb{F}_2 \to \partial \mathbb{F}_2$ be the shift map, i.e. $(w_n) \mapsto (w_{n+1})$. Note that the graph of σ is just a directing of the treeing above.



In other words, there is a secret selected end in each component of the treeing, which makes it hyperfinite.

It turns out that these secret ends are revealed by quasi-invariant measures (i.e. μ s.t. $E_{\mathbb{F}_2}$ is μ-mcp).

Let's define a quasi-inv. measure μ on $\partial \mathbb{F}_2$ using simple nonbacktracking random walk: the measure on a clopen cylinder $[a b^{-1} a^{-1} b] :=$ the set of all $w \in \partial \mathbb{F}_2$ starting with $a b^{-1} a^{-1} b$, $\mu([a b^{-1} a^{-1} b]) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$.



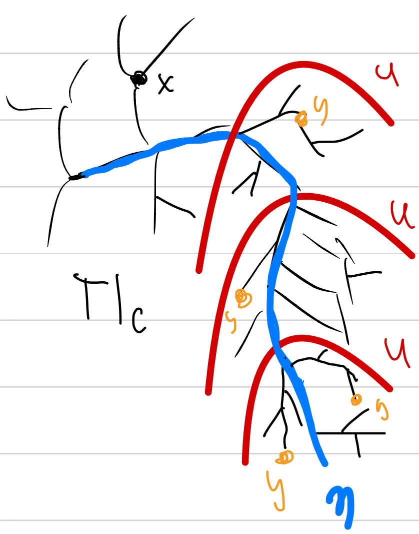
The Radon-Nikodym cocycle is given by:
$w^x(\sigma(x)) = 3$. Exercise 2. Prove this.

Vanishing ends and generalized Adams' dichotomy

Let T be a treeing of a mcp CBER E on (X,μ) with $w: E \to \mathbb{R}^{\times}$ the associated Radon-Nikodym cocycle.

Remark. The assumption of mcp on a CBER is non-restrictive because:
(a) $\exists$ a conull set $X' \subseteq X$ s.t. $E|_{X'}$ is mcp.
(b) $\exists$ prob meas. $\mu' \gg \mu$ and E is mcp wrt μ'.

Def. For a T-component C, an end $\eta \in \partial_T C$ is called vanishing if $\lim_{y \to \eta} w^x(y) = 0$, where x is any/some pt in C.
This means $\forall \varepsilon > 0$ $\exists$ "neighbourhood" U of η, i.e. sides of finite sets, $w^x(y) < \varepsilon$ $\forall y \in U$.
Thus η is nonvanishing if $\limsup_{y \to \eta} w^x(y) > 0$,
i.e. $\exists \varepsilon > 0$ $\forall$ neighbourhood U of η, $\exists$ $y \in U$ with $w^x(y) \geq \varepsilon$.



We say that η has w-finite geodesics, if each ray to η is w-finite.
(We say that $A \subseteq C$ is w-finite if $w^x(A) := \sum_{y \in A} w^x(y) < \infty$.)

Generalized Adams dichotomy (Ts–Tucker-Drob). Let E, T, w, μ be as above.

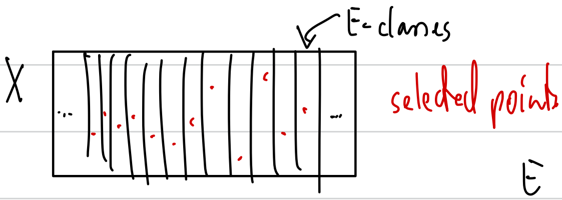
(a) E is amenable $\iff$ a.e. T-component has ≤ 2 nonvanishing ends;
$\iff$ a.e. T-component has ≤ 2 ends with w-infinite geodesics.

(b) E is μ-nowhere amenable $\iff$ a.e. T-component has perfectly-many nonvanishing ends.
(This means that the set of nonvanishing ends is nonempty closed and has no isolated pts.)
$\iff$ for a.e. T-component, the set of ends with w-infinite geodesics is nonempty G_δ and has no isolated.
(Open question. Is this set closed?)

The pmp Adams' dichotomy can be proven using cost := expected degree of graphings. But the theory of cost breaks outside of pmp. However, Adams' original prove used end selection instead, which will discuss next.
To state it, we need to define smoothness (= triviality notion for CBERs).

Smooth CBERs.

A CBER E on X is smooth $\iff$ $\exists$ Borel selector, i.e. a Borel $s : X \to X$ s.t. $x E y \iff s(x) = s(y)$ and $s(x) \in [x]_E$.



E is μ-smooth if E is smooth on a Borel conull set.

Exercise 3. An mcp E is μ-smooth $\iff$ a.e. E-class is w-finite.

Exercise 4. Smooth $\Rightarrow$ hyperfinite. (Hint: Use Feldman–Moore.)

Ben Miller showed that endless treeings are smooth. We generalize this:

Characterization of smoothness (Ts.–TD). An mcp treeing is μ-smooth $\iff$ all ends are vanishing (after discarding a null set).

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Lecture 3

End selection.

Let E be a nowhere smooth treeable cmp CBER on (X, μ) and let T be a treeing of it.

Def. For a subeq. rel. $F \subseteq E$, an F-invariant Borel end selection is an F-invariant map $X \to \mathcal{P}_{fin}(\partial X)$ mapping each $x \in X$ to a finite set $\mathcal{E}_x$ of ends in the T-connected component of x such that its lift $X \to \mathcal{P}_{fin}(\mathrm{Rays}(T))$ by $x \mapsto \{[x, \eta)_T : \eta \in \mathcal{E}_x\}$ is Borel. Here by $[x, \eta)_T$ we mean the T-ray starting from x representing η.

Theorem (Adams–Lyons; see JKL := Jackson–Kechris–Louveau).
With the above assumptions, F is amenable $\iff$ it admits an F-inv. Borel end selection of ≤ 2 ends.
If F itself is nowhere smooth, then there is a maximal such selection and it's unique up to null sets.

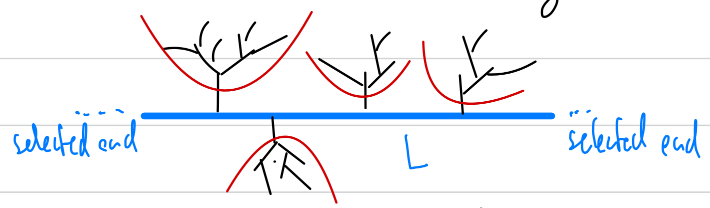
Now let's prove:

Theorem (Adams). A pmp treeing T is amenable $\iff$ it has ≤ 2 ends in a.e. connected component.

Proof-sketch. $\Leftarrow$. Just select all (≤ 2) ends in each T-component, so T is amenable.

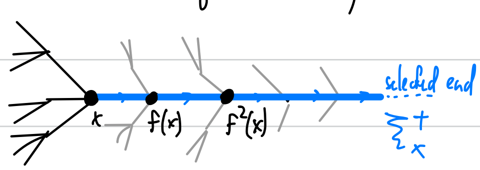
$\Rightarrow$. Let $x \mapsto \mathcal{E}_x$ be a maximal $E := E_T$-invariant Borel end selection from T.
WLOG, it's enough to consider the following two cases.

Case 2. Two ends are selected by each E-class. For each E-class C, let L be the T-line between the two selected ends.



The sides of L are w-finite (being the classes of a smooth CBER F of of being on the same side of the line, use Exercise 3). Thus, if E is pmp, each side is finite, so it has no ends; and in mcp, all ends in the sides are vanishing.

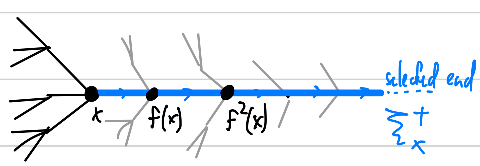
Case 1. One end is selected by each $x \in X$. Denote it by ξ_x^+. Let $f : X \to X$ be the map taking each $x \in X$ to the next point on the ray $[x, \xi_x^+)$. Thus,



T is just the undirected version of $\mathrm{Graph}(f)$. We want to show that f-back ends don't exist. Do a mass transport by letting each $x \in X$ give its whole mass (1) to each of its preimages, so each x gives $\geq 1^{*}$ and receives $= 1$ (by pmp, because $\exists$ one f-image of x). But because we selected only 1 end, f cannot be essentially 2-ended, so there are points in each E-class that have ≥ 2 preimages. This shows $2 \leq 1$, by Mass Transport, $\perp$.
(*) This is after iteratively shaving the leaves. $\square$

We will now show how to prove Case 1 in the mcp setting.

Proof of mcp Case 1.

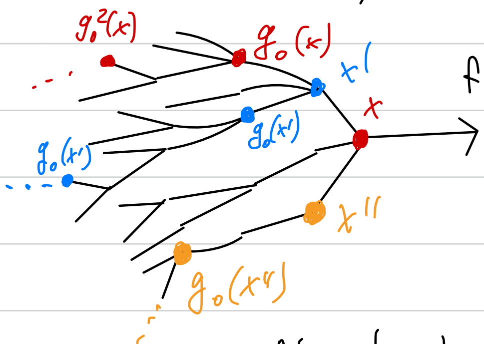


We need to show that the f-back ends are vanishing.

A technical analysis boils this down to proving the following:

Core Lemma. Almost every E-class has points $x \in X$ s.t. $\forall y$ behind x (i.e. $y \in \bigcup_{n \geq 1} f^{-n}(x)$) $w(x) > w(y)$.

We now prove the core lemma. Suppose otherwise, then WLOG, we assume that for all $x \in X$ there is a pt y f-behind x s.t. $w(x) \leq w(y)$. By Luzin-Novikov uniformization, $\exists$ Borel $g_0 : X \to X$ that maps each x to such a y.



Thus, $w(x) \leq w(g_0(x)) \leq w(g_0^2(x)) \leq \dots$
Hence, each g_0-orbit is w-infinite, so by Exercise 3, the orbit eq. rel. $\mathbb{E}_{g_0}$ is nowhere smooth (and amenable because it's given by a function).

Also by Luzin-Novikov, $\exists$ Borel $g_1 : X \to X$ taking each x to a $y \in f^{-1}(x)$ s.t. the path from x to $g_0(x)$ doesn't go through y, if such a y exists; otherwise $x \mapsto x$. Since f is not essentially 2-ended, there will be x with $|f^{-1}(x)| \geq 2$, so the orbit eq. rel. $\mathbb{E}_{g_1}$ has a nonsingleton class in each E-class. $\mathbb{E}_{g_1}$ is also amenable.

By our Paddle-ball lemma for eq. rels (applied to $\mathbb{E}_{g_0}$ and $\mathbb{E}_{g_1}$), we get that $\mathbb{E}_{g_0}$ and $\mathbb{E}_{g_1}$ are freely independent, which contradicts the amenability of E by:

Theorem (Carriere-Ghys). Let E be a CBER on (X, μ) and assume that E contains a free product $F_0 * F_1$ of eq. rels. such that:

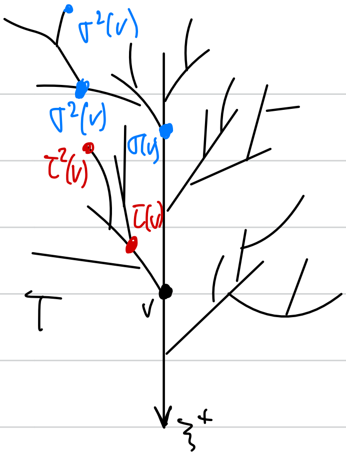
(i) F_0 is μ-nowhere smooth;

(ii) a.e. E-class contains a non-trivial F_1-class.

Then E is μ-nowhere amenable.

Although we need the more general Paddle-ball Lemma for eq. rels, I will only state it for permutations for simplicity.

Paddle-ball Lemma for permutations (Ts-TD). Let T be a tree on a vertex set V. Let ξ^+ be a distinguished end of T and let $\Sigma \subseteq \mathrm{Sym}(V)$ be s.t. $\forall v$

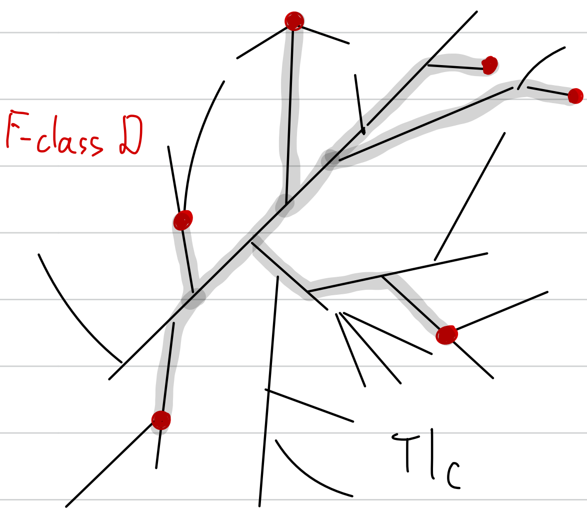


(i) $v \in [\sigma(v), \xi^+)_T \quad \forall\, \sigma \in \Sigma$.

(ii) $v \in (\sigma(v), \tau(v))_T \quad \forall\, \sigma, \tau \in \Sigma$.

Then Σ freely generates a free group $\langle \Sigma \rangle$, whose action on V is free.

Which ends are selected?



Let E be a treeable mcp nowhere smooth CBER with a treeing T and let $F \subseteq E$ be amenable.

Theorem (Ts–TD). Fix an F-class D. Let T_D be the subtree spanned by D, i.e. T_D is the T-convex hull of D. Discarding a null set, we have:

(a) When T is pmp, T_D has ≤ 2 ends and these are exactly the selected ends.

(b) In general for mcp T, T_D has ≤ 2 nonvanishing ends along D, i.e. $\limsup_{\substack{y \to \eta \\ y \in D}} w(y) > 0$, and these are exactly the selected ends.

Background reading.

Kechris has a survey on CBERs.

Also JKL := Jackson – Kechris – Louveau!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Viennese specialities